Title: Operations with Functions
Class: Math 107, Math 111, Math 120, Math 137
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Instructions to tutor: Read instructions and follow all steps for each problem exactly as given.
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Operations with Functions

Purpose:
This is intended to refresh your knowledge about operations with functions, including addition, subtraction, multiplication, division, and composition.

Just as we may perform the basic operations of addition, subtraction, multiplication, and division on real numbers, we may also do this with functions. The arithmetic of functions is analogous to arithmetic with real numbers.

Suppose that $f(x)$ and $g(x)$ are functions. Then we have the following:

Addition:	$(f+g)(x) = f(x) + g(x)$
Subtraction:	$(f-g)(x) = f(x) - g(x)$
Multiplication:	$(f \cdot g)(x) = f(x)g(x)$
Division:	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, \quad g(x) \neq 0$

So in each case, to find the output of the combined function, we just perform the usual operations with the outputs of the original functions. Let's look at an example.

Example: Let $f(x) = x^2 - 2x$ and $g(x) = 5 - 2x$. Evaluate each of the following.

(a) $(f+g)(3)$

From the above, we see that $(f+g)(3) = f(3) + g(3)$, so we just need to evaluate both f and g at $x = 3$, and then add the results.

So $f(3) =$ __________ and $g(3) =$ __________ .

Now $(f+g)(3) = f(3) + g(3) =$ __________ + __________ = __________ .

(b) $(g-f)(x)$

This time we will not be substituting a value in for x, just subtracting the two functions.

So $(g-f)(x) = g(x) - f(x) =$ ___________ − ___________ = ___________ .

Did you get $5 - x^2$? If not, check that you distributed the minus sign to all of $f(x)$.

(c) $(f \cdot g)(x)$

Similar to part (b) in that we will just be multiplying the two functions, rather than substituting a value in for x.

So $(f \cdot g)(x) = f(x)g(x) =$ ___________ $\cdot$ ___________ $=$ ___________ .

Did you get $-2x^3 + 9x^2 - 10x$? Good! Let's try one more.

(d) $\left(\frac{g}{f}\right)(0)$

Similar to part (a) in that we will be evaluating both f and g at $x = 0$, then dividing the results.

So $\left(\frac{g}{f}\right)(0) = \frac{g(0)}{f(0)} = \frac{\qquad}{\qquad} =$

Did you run into a problem here? Good, that means you identified that the denominator became zero. So in this case we would say that $\left(\frac{g}{f}\right)(0)$ is undefined, or that 0 is not in the domain of $\left(\frac{g}{f}\right)(x)$ - this is because 0 is not in the domain of f.

Ok, now try the next few on your own.

1. Let $f(x) = \sqrt{x-3}$ and $g(x) = 3x - x^2$. Evaluate each of the following. Check your answers at the end of this document.

(a) $(g - f)(7)$ (b) $(f \cdot f)(x)$ (c) $\left(\frac{f}{g}\right)(4)$

So, you have seen that the basic arithmetic operations applied to functions pretty much worked the same way they did for numbers. The next operation is a bit different, and will become more and more important as you move through your mathematics coursework. This operation is called **composition of functions**.

> The **composition of** $f(x)$ **and** $g(x)$ is given by $(f \circ g)(x) = f(g(x))$.
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Note the attention to the order of the two functions. This is because, in general, composition is not commutative; i.e., order matters.

Also, be careful not to confuse this with multiplication, as much as it looks like it.

Let's take a look at a few examples.

Example: Let $f(x) = x^2$ and $g(x) = 2x - 3$. Evaluate each of the following.

(a) $(f \circ g)(5)$

First, let's apply the definition: $(f \circ g)(5) = f(g(5))$

So it appears that the output of $g(5)$ will become the input for f. This is how composition works – we apply several functions in succession, with the output of the inner function becoming the input for the outer function.

Note that $g(5) = 2(5) - 3 = 7$.

So $(f \circ g)(5) = f(g(5)) = f(7) = (7)^2 = 49$.

(b) $(g \circ f)(5)$

This will be similar to part (a), but we are switching the order.

First apply the definition: $(g \circ f)(5) =$ ___________ .

This time the output of $f(5)$ will become the input for g.

So $f(5) =$ ___________ .

Now $(g \circ f)(5) =$ ___________ .

Did you get 47? Good! Notice that we did get different answers when the order of the functions changed.

(c) $(f \circ g)(x)$

This time, we will not be substituting a value in for x – we will be substituting the entire function g into the function f.

The definition gives us $(f \circ g)(x) = f(g(x))$.

So let's replace $g(x)$ with what it is defined to be: $f(g(x)) = f(2x-3)$

I know – this feels awkward, as you are used to just substituting numbers into a function. But we can also substitute other functions in place of the independent variable. Hang in there – you will get used to this.

What we are going to do is replace the independent variable x in the function f with the expression $2x-3$.

Now $(f \circ g)(x) = f(g(x)) = f(2x-3) = (2x-3)^2$.

(d) $(g \circ f)(x)$

This time we switch the order the functions are applied.

The definition gives us $(g \circ f)(x) = g(f(x))$.

So we need to substitute the function $f(x)$ into the function $g(x)$, wherever we see the independent variable x.

Now $(g \circ f)(x) = g(f(x)) = g(\_\_\_\_\_\_\_\_\_) = 2(\_\_\_\_\_\_\_\_\_) - 3 = \_\_\_\_\_\_\_\_\_$.

Did you get $2x^2 - 3$? Good! This operation will take some practice before you get comfortable with it. To this end, try the next few on your own.

2. Let $f(x) = 2x^2 - 3x$ and $g(x) = \dfrac{1}{x}$. Evaluate each of the following.

(a) $(f \circ g)(1/2)$ (b) $(g \circ f)(-3)$ (c) $(f \circ g)(x)$

3. Let $f(x) = x^2 - 3$ and $g(x) = \sqrt{x+1}$. Evaluate each of the following.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

(c) $(f \circ f)(x)$

(d) $(f \circ g)(4)$

(e) $(g \circ f)(1)$

(f) $(g \circ g)(8)$

Check your answers. Consult a tutor if you are having difficulties.

1. (a) -32 (b) $x - 3$ (c) $-\dfrac{1}{4}$

2. (a) 2 (b) $\dfrac{1}{27}$ (c) $\dfrac{2}{x^2} - \dfrac{3}{x}$

3. (a) $x - 2$ (b) $\sqrt{x^2 - 2}$ (c) $x^4 - 6x^2 + 6$ (d) 2 (e) undefined (f) 2