Comparison Test Notes for Improper Integrals

Suppose we have an improper integral of the form:

$$\int_a^b f(x)dx \text{ where } f(x) \geq 0 \text{ for } a \leq x \leq b$$

We can use a comparison test to check for convergence or divergence by finding a function that is always larger or smaller than $f(x)$ when $a \leq x \leq b$

Testing for Divergence:
Find a function $g(x)$ so that

$$0 \leq g(x) \leq f(x) \text{ when } a \leq x \leq b$$

$$\text{and } \int_a^b g(x)dx \text{ is } \underline{\text{divergent}}$$

Since $\int_a^b g(x)dx$ is divergent, the <u>larger</u> integral $\int_a^b f(x)dx$ must also diverge.

Testing for Convergence:
Find a function $g(x)$ so that

$$f(x) \leq g(x) \text{ when } a \leq x \leq b$$

$$\text{and } \int_a^b g(x)dx \text{ is } \underline{\text{convergent}}$$

Since $\int_a^b g(x)dx$ is convergent, the <u>smaller</u> integral $\int_a^b f(x)dx$ must also converge.

Common Functions to Test for Convergence or Divergence

$\int_a^b \frac{1}{x^n} dx$ is easy to integrate, so it's very useful in comparison tests. Convergence and divergence depend on the values we use for a, b and n.

If $n = 1$, then

$$\int_a^b \frac{1}{x} dx = \ln(x)\Big|_a^b$$

If $n \neq 1$, then:

$$\int_a^b \frac{1}{x^n} dx = \frac{-1}{(n-1)x^{n-1}}\Bigg|_a^b$$

	$\int_0^b \frac{1}{x^n} dx$	$\int_a^\infty \frac{1}{x^n} dx$
$n = 1$	Divergent	Divergent
$n > 1$	Divergent	Convergent
$n < 1$	Convergent	Divergent

$\int_a^b \frac{1}{e^{nx}} dx$ is also useful, evaluate the integral yourself to confirm the results in the table.

	$\int_{-\infty}^b \frac{1}{e^{nx}} dx$	$\int_a^\infty \frac{1}{e^{nx}} dx$
$n < 0$	Convergent	Divergent
$n > 0$	Divergent	Convergent